# Time at Home,
# and the Inexorable Rise of the Chess Block

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**Abstract**: The "chess block" is an intuitive design for the urban block. It maximizes maximally daylit houses on the block. We here address the chess block because it appears to increasingly dominate contemporary construction. The chess block maximizes the block's daylit houses, daylit windows, and even profit. It is the built manifestation of the large developer's plan. Developers of mere sub-divisions of the block often fail to coordinate as successfully. But when households have more time-at-home (so that they value daylighting more), that failure becomes more acute. Sub-developers then coordinate more. Because the chess block is less dense, its increasing adoption pushes the marginal resident out, and bright houses' price up. Households' increasing time-at-home offers a novel, simple, secular and global explanation of: the secular, global rise of the chess block and the price of housing, and of growing collusion in construction.

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# 1 Introduction

Anyone is familiar with the chess board. Its design simply "colors in" the squares of the board in black and white such that no two adjacent squares share the same color.

If we read the board as an urban block, a black square as a house, and a white square as a yard, we have this paper's "chess block". Intuitively, the chess block not only (i) fully *daylights* each of its houses, but also (ii) *packs* as many of those fully daylit houses as possible into the block. Not only is the chess block "bright", providing each of its houses with daylight from all around; also it is "dense", maximizing maximally bright houses on the block. Fig. (1a) illustrates the chess block on a $6 \times 5$ example. Fig. (2) shows a chess block under construction in Dortmund, Germany. Levittown (Fig. (3b)) is a chess block in a city periphery. But Stuyvesant Town (Fig. (3a)) and the blocks in Guangzhou (Fig. (3c)) and Munich (Fig. (3d)) are chess blocks near city centers, too.

In this paper we explore the properties, the origin, and the future of the chess block. We set out the many ways in which the chess block creates opportunities for daylighting the home. We prove that the chess block is a large developer's best design. We show how and when the chess block emerges in sub-developer equilibria, too. We then offer a secular, global explanation that the secular, global rise of the chess block calls for. Households enjoy more time-at-home today than in the past. So they are willing to pay more for a daylit home. The real estate industry responds by building chess blocks' bright houses. Yet while chess blocks daylight houses well, they push marginal residents out. In short, more time-at-home contributes to the secular, global rise in house prices.

At first sight the chess block seems a natural, if not outright appealing design. How else would a developer organize his houses on a given block? But let us early on reverse the vantage point here. Not only is the chess block just one design among many. Even for the number of parcels small (equal to 30 say), the number of distinct designs is *vast* (i.e. 1.073 billion app.). For instance, there are $\binom{30}{20}$ different ways alone to build 20 houses into a block of 30 parcels. Also, a block is not necessarily built by *a* developer. Often multiple smaller, or "sub-", developers compete on the block. We should expect these sub-developers to have little to no regard for the daylight neighboring sub-developers enjoy. Sub-developers have no shared interest in making the block "appealing".

Historically, block designs rarely strike the chess block's delicate balance between lighting and packing. Fig. (1b) shows the dense "Haussmann-type" block often still standing today, say, in Paris' 8th arrondissement, Barcelona's Eixample, New York's Upper West side or Berlin's Kreuzberg district. Fig. (1c) resembles the "perimeter block" typical of Georgetown/Wash. or of many other US historical city centers. Fig. (1d) shows the orderly "rows block" popular in Germany's Weimar Republic. To varying degrees, all of these blocks sacrifice daylight exposure for being able to pack more houses into the block. In short, urban blocks built in the past typically are both: dense and dark.

Those dense, dark blocks are no longer built today. Instead, contemporary construction appears to converge on the chess block. One theoretical result below is that the chess block maximizes the number of windows that are daylit. Some random real block is unlikely to precisely attain the chess block's maximum daylit windows. But we do not argue here that the resulting deviation between the observed number of windows and its maximum is small. Instead we suggest that it has become *smaller* (and is likely to get smaller still). An empirically minded reader may want to see the evidence here. A study of global trends in urban design is desirable. Yet even absent that study (we are aware of none), four reasons seem compelling enough for inspecting the chess block *now*:

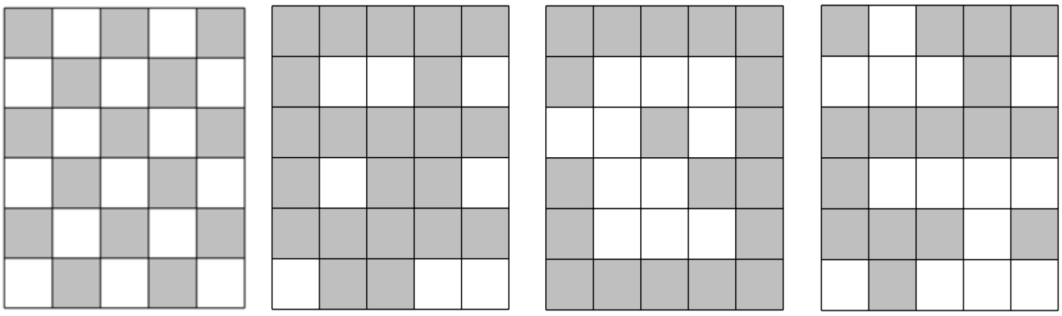
First, the anecdotal evidence on chess block proliferation speaks for itself. Chess block construction virtually all but surrounds us. It seems the rare contemporary urban block that is *not* both bright and dense. (We invite the city-based reader to visit ongoing block developments in her or his neighborhood.) Second, as a matter of caution, the chess block should be understood now, not later. An early understanding of the chess block's role for welfare can help us assess ways to improve on it. Third, much can be learnt from the chess block's seeming simplicity. The chess block suggests tools that can benefit our understanding of urban design generally. Fourth, only a mature theory of the chess block supplies the meaningful (non ad hoc) propositions that can inform data consultation.

Why has the chess block become so dominant in today's construction? To what effect? Who are its builders? Architectural historians and urban planners point to the legacy of early modernist architects like Louis Sullivan and Frank Lloyd-Wright, full-on modernists such as Le Corbusier, or *Bauhaus* and Harvard architects like Walter Gropius and Marcel Breuer (e.g. Goldberger (2023)). Their modernist design ideas are echoed by the chess block. Le Corbusier's demands for "light and air for all" literally translate into the block being both daylit throughout ("light and air") and packed ("for all"). In this paper we offer very different, and new, answers to the questions raised above. These derive from an economics perspective instead. They appeal (i) to long run changes in societal constraints (i.e. the decline in the household's time at work) and (ii) to the real estate industry response they induce (i.e. sub-developer coordination on block design).

We first offer a parsimonious economic model of the chess block. As we will see from a variety of different perspectives, the chess block makes the most efficient use of block daylight. We prove, highlight and acknowledge four core admirable properties of the chess block. The chess block (1) minimizes the number of yards necessary to daylight every house, (2) maximizes the number of fully daylit houses, (3) maximizes the total number of daylit windows, and (4) maximizes block profit (for a reasonable range of prices). Large, or "block-", developers negotiate the block's daylighting externalities best when building the chess block. The chess block is the built manifestation of a large developer's optimizing plan, elegantly maximizing all within-block daylight externalities.

Multiple smaller sub-developers find it difficult to coordinate on the chess block. Any individual sub-developer has little incentive to avoid occluding neighboring sub-divisions' houses. The individual sub-developer is tempted to overbuild. Especially near the city center, the great location value of any house makes dense blocks likely. In the city periphery, smaller location value allows sub-developers to coordinate on the chess block. There sub-developers tacitly collude. Their sub-division designs amalgamate into the chess-block. In the periphery, the chess block (5) also emerges as an equilibrium configuration. This adds to the chess block's efficiency properties (and cautions against reading developer competition off observed block design or real estate industry concentration).

Here a microeconomic analysis of urban daylight comes into view. (Oddly, urban daylight, or the urban window that lets daylight into the house, has received little attention in urban economics.) Departing from a simple "axiomatic" approach to daylighting technology, equilibrium specifies the design of the block, for any space-time-context one may encounter within the city and for a variety of land ownership constellations. A



(a) **“chess block”** 15 houses, 15 yards 60 daylit windows

(b) **“Haussmann block”** 22 houses, 8 yards 34 daylit windows

(c) **“perimeter block”** 19 houses, 11 yards 42 daylit windows

(d) **“rows block”** 16 houses, 14 yards 38 daylit windows

Notes: (i) The figure’s block designs are shown on a $6 \times 5$ block grid. (ii) Fig. (1a) shows the generic “chess block”. The chess block starts with a house on the block’s “top left corner”, then successively alternates built-up parcels (houses) with parcels not built up on (yards). This pattern is easily extended to any $m \times n$ grid. (ii) Fig. (1b) shows an “Haussmann block” that is nearly built up on fully. (iii) Fig. (1c)’s “perimeter block” builds up on the block’s perimeter, but essentially maintains an open block interior. (iv) Fig. (1d) shows the “rows block”.

Figure 1: The chess block, and three historical alternatives

fine-grained map of daylightings and occlusions emerges. That map assigns every house its individual daylighting quality. In this context we experiment with a secular, near-global increase in households’ valuation of daylight in the home that is suggested by the secular, near-global reduction in households’ time-at-work reported in the literature.

Four long-run adjustments then become “inexorable”: Those are adjustments in (i) real estate industry organization, (ii) urban block design, (iii) population density, and (iv) the price of residential real estate. Sub-developers extend their collusion to blocks even nearer to the center. Non-cooperative, i.e. denser and darker, blocks become fewer, and the rise of the chess block becomes inevitable. To accommodate the city’s exogenous population, city rings must now extend further out, pushing the marginal resident further away from the city center. Thus bright houses’ prices rise. And while it is true that dark houses’ prices fall, dark houses also recede. When time-at-home has fallen enough, all developers collude not to occlude. Chess blocks will be ubiquitous. Cheap dark houses will have disappeared. Only bright – and expensive – houses remain.

Occluding a neighboring window is the quintessential externality. (Where the occluded house foregoes one daylit window, the occluding house may gain three daylit windows.) So on the one hand, internalizing the occlusion externality is really only one more instance of property rights centralization (Coase (1960)). On the other hand, making explicit the positive daylight externality joint with its twin, the negative occlusion externality, fills in desirable detail. This detail shows up in the optimal level of the occlusion externality (it is 0), in our predictions of the spatial remit of chess block dominance (in the periphery, and not in the city center), in externality internalization’s long-run impact on house prices (prices rise) and on the spatial structure of the city at large (growing shades in private open space and receding shade in public open space).

No microeconomic model of urban daylight has, to the best of our knowledge, been put forward to date. Ideally, any such model builds on the rich existing insights in the literatures on developer interaction, daylight valuation, industry consolidation, house prices, land assembly, and block design. Strange (1992) is an early exploration of externalities

Notes: (i) The figure shows a block currently under construction (not far from the center of the medium sized city of Dortmund, Germany). (ii) This block resembles, and hence may be counted as an example of, the paper's generic "chess block" in Fig. (1a)). (iii) The block is built by a single developer, "Assmann Group". (iv) A number of pictures on construction progress are made available, by the developer, on the developer's website.

Figure 2: An example of contemporary residential real estate

across adjacent neighborhoods among interdependent developers. Here any individual developer's choice of his neighborhood housing fails to account for that housing's negative effect on neighboring housing. These developers' failure parallels the disregard our paper's sub-developers profess for neighboring sub-divisions' daylight.

Huberman/Minns (2007, Table 3) document annual hours at work between 1870 and 2000 for the US, Canada, and a subset of Western European countries. For example, for the US they find that annual hours at work declined from nearly 3,000 to less than 2,000 from 1900 to 2020. Starting from slightly higher levels, France and Germany saw their hours at work decline by even more. We expect similar trends for many other industrialized countries, too. We also suggest that all the extra time away from work is in part also spent at home. Sharkey (2024) gives recent evidence for U.S. adults. Over the period between 2003 and 2022 alone (i.e. not just since Covid-19), U.S. adults have increased their time-at-home by 1 hour and 39 minutes.

More time-at-home should have the household appreciate daylight more. Fleming et al. (2018) are the first to provide hedonic estimates of households' valuation of sunlight. Following these authors, and on data for Wellington (New Zealand), every extra hour of sunlight exposure raises the value of local real estate by 2.4%. This result does not assert that the valuation of daylight equals that of sunlight, or even that daylight valuation is greater today than it was in the past. But it does suggest that daylight consumption in the home has significant value today. Dantzig/Saaty (1973) famously proposed a compact city design that would have stacked large slices of dark housing on top of each other. Fifty years on it seems difficult to even only imagine their dark housing part of a "liveable urban environment" (as claimed by that study's title).

A large literature documents the rise in the price of housing (both owner-occupied and

(a) Stuyvesant Town


(b) Levittown, Pennsylvania

(c) Guangzhou (China)


(d) Domagkpark (Munich/Germany)

Notes: (i) The figure shows the figure ground plans of four blocks that approximate the stylized chess block in Fig. (1a), and hence in turn benefit from an understanding of the generic chess block. (ii) The blocks “Stuyvesant Town” in New York and “Domagkpark” in Munich/Germany are centrally located, Levittown and Guangzhou are at their cities’ respective urban peripheries. (iii) Source: Google Earth/Google Maps.

Figure 3: Approximate chess blocks

rented) on an almost global scale, and for almost half a century now (e.g. Knoll et al. (2014), Jorda et al. (2019), Lyons (2025) for the US). A trend towards ever stricter zoning is often suspected to drive the price of housing up (e.g. Glaeser/Ward (2008), Gallagher et al. (2024)). A landlord majority zones for low-density housing, minimum lot sizes, or excessive environmental regulations. d’Amico et al. (2024) argue that developers build less because project size falls. But growing collusion among developers may reduce housing supply, and increase house prices, too. Tacitly colluding developers embrace the chess block not because they have to, but *because they want to.* The chess block enforces a scheme of “internal” or “voluntary” zoning. When zoning is not binding, then a policy of zoning less may fail to unleash housing supply (Diamond et al. (2024)).

This paper is in seven sections. Section 2 presents the microeconomics of daylight. Section 3 sets out the chess block and its core properties. Section 4 analyzes equilibrium configurations when sub-developers share in, and compete on, the block. Section 5 analyzes the paper title’s rise in time-at-home. Section 6 allows for various extensions, and section 7 concludes. (Remarks give auxiliary results, propositions state main results.)

## 2 A Closed Daylit City

The closed city model represents an entire urban system of like-minded cities across which households are mobile (Brueckner (1987)). Our analysis is set into the context of a closed *daylit* city. This is, with the exception of Dascher/Haupt (2025), a novel city model. An area of land is large enough for our representative city to unfold inside it. We partition the city's area into rectangular *blocks* by horizontal and vertical strips of land (of one unit width) called *streets* and *roads*, respectively. A block is an $m$ by $n$ array of $mn$ equal-sized square parcels, also called *lots*. The city center is coincident with the intersection of some street with some road. All jobs, shops, and schools are found in the center. City residents must commute there to avail of them. Within-block travel is free. So commutes along the street grid need only depart from, and terminate at, the corner of the resident block nearest to the center. Manhattan type distance $r$ equals $n+1$ times street sections plus $m+1$ times road sections travelled (Yinger (1993)). Commuting a distance of $r$ twice a day costs $tr$. City population is $L$.

Depending on whether its (at most four) neighboring lots are built up on or not, each lot may be *daylit* not at all, or once, twice, three or even four times. Five specific assumptions will govern any lot's daylight reception. First, only indirect daylight, deflected by the lot's neighboring yards, daylights that lot. Direct daylight, streaming in from the sky, does not. Second, daylight deflected into any given lot only streams in from as far as that lot's neighbors, and never from lots further afar. Third, any yard daylights all four neighboring lots just as well as it daylights a single built-up neighbor. Fourth, sunlight streaming into any lot from different directions "adds up". Alternatively justified, it has the lot enjoy sunlight longer. And fifth, facing south – or north on the Southern hemisphere – is as good as facing any other direction. Our discussion assumes a house of $h = 3$ or 4 stories, but even 5 or 6 stories seem defensible, too.

To live in the city, the household must live in a house. Utility is $e + wv$, where $e$ is food, $w \in \{0, \ldots, 4\}$ is the number of the house's daylit *windows*, and $v$ is the household's *valuation* of the daylight streaming into any of those daylit windows, $v > 0$. Valuation $v$ is increasing in the household's time-at-home. When $w$ is 4, the house is *fully daylit*; while when $w$ is 0, it is *dark*. Construction cost is zero. Let $p(r)$ denote the rent of a dark house at distance $r$. Thus a house with $w$ daylit windows rents out at $p(r) + wv$.

A resident of a fully daylit house at the city boundary $\tilde{r}$ (where an abundance of fully daylit countryside houses is just within a step's reach) eats whatever remains of the wage after taking $t\widetilde{r}$ off. At the same time, that resident, when living in a dark house at distance $r < \tilde{r}$, only enjoys an amount of food equal to the wage minus the cost of living $\big(p(r) + tr\big)$ and minus the daylighting disadvantage $4v$. So in equilibrium,

$$p(r) \;=\; -4v \;+\; t\big(\widetilde{r} \,-\, r\big) \qquad\qquad \text{(dark rent).} \qquad (1)$$

expresses dark rent at distance $r$ and daylight valuation $v$ and for city size $\widetilde{r}$. For city size given and with a secular rise in daylight valuation, dark urban houses ($w = 0$) *lose value*; then only bright urban houses ($w = 4$) are able to maintain their value.

Via eq. (1), in large cities ($\tilde{r}$ large) or, for that matter, anywhere households enjoy little time-at-home ($v$ small), dark rent can be expected to be positive at least near the center ($r$ small). Historically, dark rent often was positive at the center, else New York's "old law tenements" would not have multiplied (Glaeser (2012)). However, in any average size city today, we expect negative dark rent even at the city center, i.e.

$$p(0) \;<\; 0 \qquad\qquad \text{(negative dark rent)} \qquad (2)$$

or $t\widetilde{r} < 4v$. For example, consider "Munger Hall", a recent student dormitory design for Berkeley, California. Munger Hall was to have mostly windowless apartments. Tellingly, it never got built, not even in expensive California (Cramer (2021)). (Urban design in mega-cities ($\tilde{r}$ large) or in poor countries ($v$ small) is outside the remit of this paper.)

It is $\boldsymbol{X}$ the $m \times n$ layout of the rectangular city block. Lots are labeled $(j, k)$ (or only $jk$), with row index $j$ ranging from 1 to $m$ and column index $k$ ranging from 1 to $n$. Vector $\boldsymbol{x}$, with

$$\boldsymbol{x}' = (x_{11}, \ldots, x_{1n}, \ \ldots \ , x_{m1}, \ldots x_{mn}) \qquad \text{(configuration or block)}, \tag{3}$$

translates the block layout $\boldsymbol{X}$ into the $mn \times 1$ *design* or simply *block* $\boldsymbol{x}$, of *houses* ($x_{jk}$ is 1) and *yards* (0). Any house has one *window* on each of its 4 *walls* or *faces*. Whether or not a window is more than a mere sheet of glass hanging down a wall depends on how the land on the lot adjacent to it is used. If the neighboring lot is a yard, daylight will stream in through the window pane; while if that lot has a house built onto it, no daylight will. Let $\boldsymbol{\iota}$ be an $mn \times 1$-vector of 1's. Then the block has $\boldsymbol{\iota}'\boldsymbol{\iota}$ (or $\nu$ for short) lots, $\boldsymbol{\iota}'\boldsymbol{x}$ (or $N$) houses, $4\boldsymbol{\iota}'\boldsymbol{x}$ windows, and $\boldsymbol{\iota}'(\boldsymbol{\iota} - \boldsymbol{x})$ yards.

If two houses neighbor one another, they mutually *occlude* each other. Any adjacency produces *two* occlusions. Symmetric $mn \times mn$ adjacency matrix $\boldsymbol{A}$ has a 1 in row $jk$ and column $pr$ if lots $jk$ and $pr$ are adjacent, and a 0 else. There are $\boldsymbol{\iota}'\boldsymbol{A}\boldsymbol{\iota}/2$, or $\varepsilon$, adjacencies altogether. Total occlusions on the block amount to $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$. To account for daylight streaming into a lot from outside the block, we define the $mn \times 1$ vector $\boldsymbol{f}$ of *(street-)front* valuations. Whenever $f_{jk} \neq 0$, lot $jk$ is a *frontage lot.* If $f_{jk}$ equals 2, lot $jk$ is a *corner* lot, while if $f_{jk}$ is 1, lot $jk$ is a non-corner frontage lot. For *interior* lots $jk$ remaining, $f_{jk}$ is 0.

We turn to the subset of windows that are daylit. It is $\boldsymbol{x}'\boldsymbol{A}(\boldsymbol{\iota}-\boldsymbol{x})$ the number of windows daylit from within the block. These windows are "internal daylightings" or "internally daylit windows", $\Lambda_i$ for short. Since every window facing the street is daylit, it is $\boldsymbol{x}'\boldsymbol{f}$ the number of windows daylit from outside the block. Those windows are "external daylightings" or "externally daylit windows", denoted $\Lambda_o$. In short,

$$\Lambda_i(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{A}(\boldsymbol{\iota} - \boldsymbol{x}) \qquad \text{(internally daylit windows)} \tag{4}$$

$$\Lambda_o(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{f} \qquad \text{(externally daylit windows)}. \tag{5}$$

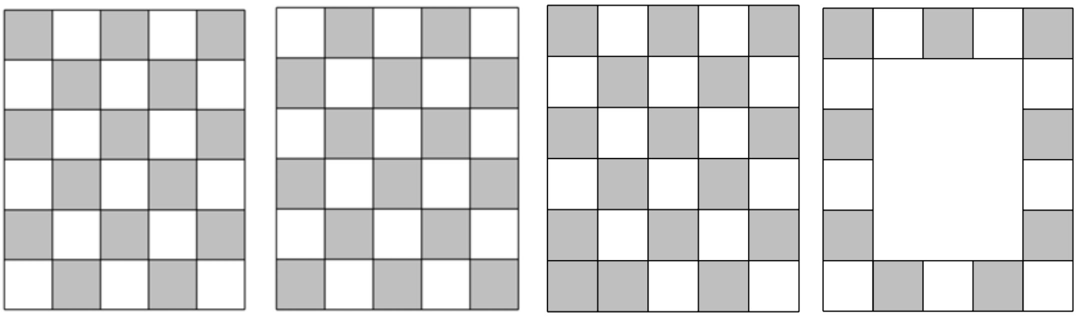
Total daylit windows $\Lambda(\boldsymbol{x})$ then are

$$\Lambda(\boldsymbol{x}) = \Lambda_i(\boldsymbol{x}) + \Lambda_o(\boldsymbol{x}) \qquad \text{(daylit windows) or} \tag{6}$$

$$\Lambda(\boldsymbol{x}) = 4\boldsymbol{\iota}'\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \qquad \text{(daylit windows)} \tag{7}$$

The second equation represents total daylit windows by subtracting occluded windows ($\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$) from total windows ($4\boldsymbol{\iota}'\boldsymbol{x}$). This is an alternative perspective very useful later.

Daylight is valuable, and profitable, only if it is consumed. A daylighting involves *two* lots: one that emits daylight, another that receives it. So while "daylit windows" and "daylightings" are synonymous below, only the term "daylighting" gives due credit to the complementarity of the window with the yard it looks onto. This complementarity suggests we focus on pair-wise adjacencies between, rather than on, lots. Translating the block's adjacency matrix $\boldsymbol{A}$ into a *graph* provides for precisely that focus. (For any graph concept introduced below see Diestel (2017) or Bondy/Murty (2018).) Let us represent lot $jk$ by *vertex* $(j, k)$. Any two vertices $jk$ and $pq$ are linked by an *edge*



(a) **block $\boldsymbol{b}$** $\Lambda_i(\boldsymbol{b}) = 49 = \varepsilon$ (b) **block $\boldsymbol{a}$** $\Lambda_i(\boldsymbol{a}) = 49 = \varepsilon$ (c) **block $\boldsymbol{d}$** $\Lambda_o(\boldsymbol{d}) = 13$ (d) **$\boldsymbol{b}$'s frontage** $\Lambda_o(\boldsymbol{b}) = 11$

Notes: (i) Houses are shown in black, yards in white. (ii) Figs. (4a) and (4b), show the two alternating designs, $\boldsymbol{b}$ and $(\boldsymbol{\iota} - \boldsymbol{b}) = \boldsymbol{a}$ on the $6 \times 5$-block, respectively. (iii) Figs. (4c) and (4d) highlight the corresponding sets of the block's frontage lots, joint with the subsets of houses and yards in them.

Figure 4: Bipartite designs

$((jk), (pq))$ whenever they are neighbors according to $\boldsymbol{A}$. Vertex set $V$ and edge set $E$ constitute *graph $G$*, with $\nu = |V| = mn$ vertices and $\varepsilon = |E|$ edges.

A design $\boldsymbol{x}$ partitions the vertex set $V$ into two subsets, i.e. into the subset of house vertices (houses for short) and that of yard vertices (yards). A daylighting really is an edge of the graph that links a house to a yard; while an occlusion is an edge that links a house to a house. From this perspective, it is clear that the block's capacity for daylightings internal to it, $\Lambda_i$, is limited by the graph's total of $\varepsilon$ edges. There can be never more internal daylightings than there are edges to $G$, $\varepsilon$ (Remark 1 (i)).

A graph is *bipartite* or *bi-colored* if there is a partition of its vertex set $V$ into two subsets such that every edge links to a vertex from either subset. Now, $G$ *is* bipartite. To see this, we define

$$B \;=\; \{(j,k) : j,k \text{ both odd or both even}\} \qquad \text{(partite set } B\text{)}. \tag{8}$$

Correspondingly, $B^c$ is the set of pairs of indices so that one index is odd and the other is even. Because $(B, B^c)$ is a bipartition of $V$, $G$ is bipartite (Remark 1 (ii)). Bipartition $(B, B^c)$ induces bipartite design $\boldsymbol{b}$, by setting $b_{jk} = 1$ if $jk \in B$, and $b_{jk} = 0$ else. The only other bipartite design is $\boldsymbol{a} = \boldsymbol{\iota} - \boldsymbol{b}$; it is induced by $B^c$. Both designs $\boldsymbol{b}$ and $\boldsymbol{a}$ sucessfully turn every adjacency "on", i.e. into a daylighting (Remark 1 (iv)).

When $\nu$ is odd (and only then), designs $\boldsymbol{b}$ and $\boldsymbol{a}$ disagree on the number of houses and windows. Here $\boldsymbol{b}$ always has 1 house, and 4 externally daylit windows, more than $\boldsymbol{a}$ (Remark 1 (iii)). This we understand by consulting the two designs' corner lots. Corner lots are daylit twice. If $\nu$ is odd, $\boldsymbol{b}$ builds up on all four corners, while $\boldsymbol{a}$ builds up on none of them. So design $\boldsymbol{b}$ has 4 external (and also total) daylightings more than $\boldsymbol{a}$.

**Remark 1 (Bipartite Designs)**

(i) *Internal daylightings are bounded by the number of graph $G$'s edges, $\Lambda_i(\boldsymbol{x}) \leqslant \varepsilon$.*
(ii) *Graph $G$ is bipartite, with $B$ and $B^c$ its partite sets.*
(iii) *Design $\boldsymbol{b}$ has 1 house, and 4 daylit windows, more than $\boldsymbol{a}$ iff $\nu$ is odd.*
(iv) *Designs $\boldsymbol{b}$ and $\boldsymbol{a}$ both have maximum internal daylightings, $\Lambda_i(\boldsymbol{b}) = \varepsilon = \Lambda_i(\boldsymbol{a})$.*
(v) *Any path between a vertex (lot) in $B$ and another in $B^c$ has odd length.*

**Proof of Remark 1**: (i), (iii) and (iv) in text. (ii) Let vertex $jk \in B$, i.e. with $j, k$ either both odd or both even. Any neighboring vertex adds 1 to, or subtracts 1 from, one, and just one, of vertex $jk$'s two indices. If both indices are odd initially, now one index is even; if both indices are even initially, now one is odd. Thus every neighbor of $jk \in B$ is in $B^c$. Similar reasoning applies to each vertex in $B^c$. So $(B, B^c)$ is a bipartition. □

(ii) Any path of even length starting with a vertex in $B$ terminates in $B$. □

We next introduce our title design, i.e. the *chess block*. In the block context, the chess block $\boldsymbol{c}$ coincides with $\boldsymbol{b}$ if $\nu$ is odd, and with either $\boldsymbol{b}$ or $\boldsymbol{a}$ if $\nu$ is even (Definition 1). Up to that ambiguity, the chess block is well defined. The set of the chess block's houses $C$ coincides with $B$ if $\nu$ is odd, and with either $B$ or $B^c$ if $\nu$ is even. In our definition, the chess block is that bipartite design that always has at least as many houses as the other. Fig. (1a) illustrates the chess block for when $\nu$ is even. In our $6 \times 5$ example, the chess block has $\Lambda_i = \varepsilon$ or 49 internal daylightings and $\Lambda_o = 11$ external daylightings.

**Definition 1 (Chess Block)**
*Chess block* $\boldsymbol{c}$ *is* $\boldsymbol{b}$ *or* $\boldsymbol{a}$ *if* $\nu$ *is even, and* $\boldsymbol{b}$ *if* $\nu$ *is odd.*

We now let a *block developer* own, and develop, all of the block's lots. The aggregate value of dark houses is $p\boldsymbol{\iota}'\boldsymbol{x}$, or $pN$ (and negative). The aggregate worth of daylit windows is $v\Lambda(\boldsymbol{x})$ (and positive). The block developer maximizes block profit, $\Pi(\boldsymbol{x})$,

$$\Pi(\boldsymbol{x}) \;=\; p\boldsymbol{\iota}'\boldsymbol{x} \,+\, v\big(\Lambda_i(\boldsymbol{x}) + \Lambda_o(\boldsymbol{x})\big) \quad \text{(block profit)} \tag{9}$$

by optimally choosing $\boldsymbol{x}$. (Dropping structures while keeping windows unfortunately is not feasible.) There is a large number of distinct possible designs $\boldsymbol{x}$ to choose from. That number is equal to $2^\nu$, and it is exponentially increasing in $\nu$.

We now break down the block developer's original (non-linear, integer) program (9) into two smaller steps. We first recognize that the block developer could start by maximizing daylit windows $\Lambda$ via choosing the optimum design $\boldsymbol{x}$ for parametric $N$. The value function to this problem is the block's *daylight frontier* $\Lambda(N)$, in eq. (10).

Once the daylight frontier is found, second, the block developer chooses the number of houses $N$ that maximize profit along it. This then yields maximum profit, in eq. (11):

$$\Lambda(N) \;=\; \max_{\boldsymbol{x}} \big(\Lambda_i(\boldsymbol{x}) \,+\, \Lambda_o(\boldsymbol{x})\big) \text{ s.t. } \boldsymbol{\iota}'\boldsymbol{x} = N \text{ (daylight frontier).} \tag{10}$$

$$\Pi(p, v) \;=\; \max_{N} \big(pN \,+\, v\Lambda(N)\big) \qquad \text{(maximum block profit).} \tag{11}$$

This two step procedure yields the optimum number of houses $N(p, v)$, the optimum number of daylit windows $\Lambda(p, v)$, the optimal block design $\boldsymbol{x}(p, v)$ and maximum block profit $\Pi(p, v)$. (For a brief formal exposition see the Appendix.) Where the second step merely involves linear optimization, the first step requires us to solve a non-linear program.

## 3 The Chess Block

We first show that the chess block fully daylights all of its houses by a minimum set of yards (Proposition 2). This perspective provides a lower bound on the maximum number of daylit windows. We then show that the chess block even maximizes daylit windows (Proposition 3). This sets up a first point on the daylight frontier $\Lambda(N)$, from which further points follow. We go on to prove that the chess block maximizes block profit (Proposition 4), and that it does uniquely so. The chess block is the *only* profit maximizer (Proposition 5). It unfailingly guides a large developer into maximum profit. The chess block's physical simplicity coincides with its economic profitability.

A *cover* of the graph is a subset of vertices such that every edge links to a vertex in that set. If the set of yards is a cover, every window is daylit. Clearly the chess block's set of yards, $C^c$, is a cover (Proposition 1 (i)). Next, a *matching* is a subset of edges no two of which are incident with the same vertex. Fig. (5b) shows a matching $M^*$ that pairs successive vertices. A total of $\nu$ ($\nu - 1$) vertices are used up in the process if $\nu$ is even (odd). Frankly, it is impossible to have more matches than $\nu/2$ (($\nu - 1)/2$) if $\nu$ is even (odd). It is $M^*$ a maximum matching.

**Proposition 1 (Chess Block has Minimum Yard Cover)**

(i) *Chess block yards are a cover.*
(ii) *Graph G's maximum matching has $\nu/2$ edges if $\nu$ is even, and $(\nu - 1)/2$ if $\nu$ odd.*
(iii) *The chess block's yards $C^c$ are a minimum cover.*
(iv) *The chess block's houses $C$ are the maximum set of fully daylit houses.*

**Proof of Proposition 1**:

(i) $(C, C^c)$ is a bipartition: Every edge links to vertices in both $C$ and $C^c$. □

(ii) Generally, $|M| \leqslant \nu/2$. Consider the set $M$ of independent edges $((1,1),(1,2))$, $(1,3),(1,4)$, etc. (Fig. (5b)) As it has $\nu/2$ edges, $M$ is a maximum matching. □

(iii) By Kőnig's theorem, a minimum cover must have $\nu/2$ vertices. Yet house cover $|C|$ attains $\nu/2$ when $\nu$ is even. (For $\nu$ odd, $|C| = (\nu - 1)/2$.) □

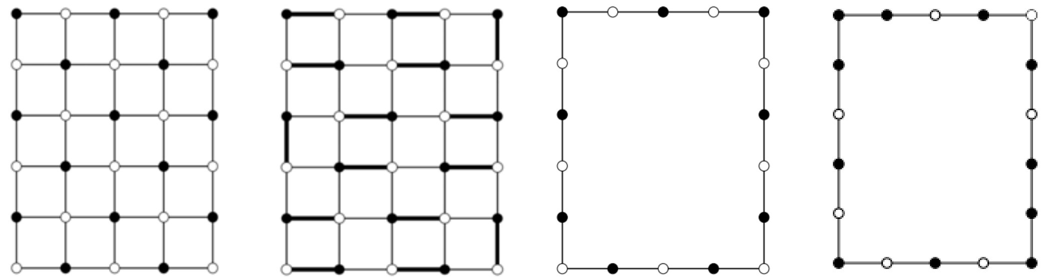
(iv) Suppose there are more than $|C|$ fully daylit houses. But then there are less than $|C^c|$ yards. By Proposition 1 (iii), those yards cannot be a yard cover. □

A famous theorem on bipartite graphs (i.e. Kőnig's theorem) informs us that any yard cover, that has as many vertices as $M^*$ has edges, is minimum. But the chess block's set of yards $C^c$ achieves just that. It has $\nu/2$ yards if $\nu$ is even, and $(\nu - 1)/2$ yards if $\nu$ is odd. So the chess block's set of yards is a minimum cover. This has an immediate implication for daylit houses. If the chess block's yards $C^c$ are a minimum cover, then the chess block's set of houses $C$ is the maximum set of fully daylit houses. There cannot be more fully daylit houses than with the chess block (Proposition 1 (iv)).

This yields a preliminary result: If the block always allows for $\nu/2$ fully daylit houses if $\nu$ is even, and for $(\nu+1)/2$ fully daylit houses if $\nu$ is odd, then 4 times these figures indicate lower bounds on the maximum number of daylit windows in the block, $\max \Lambda(\boldsymbol{x})$:

$$\max_{\boldsymbol{x}} \ \Lambda(\boldsymbol{x}) \ \geqslant \begin{cases} 2\,\nu & \text{if } \nu \text{ is even} \\ 2(\nu+1) & \text{if } \nu \text{ is odd} \end{cases} \tag{12}$$

While there cannot be more fully daylit houses than with the chess block $\boldsymbol{c}$, we cannot rule out that daylit windows may be more with other configurations $\boldsymbol{x} \neq \boldsymbol{c}$. We need to address daylit windows directly. We recall that the chess block exhausts all potential daylightings from within the block, $\Lambda_i = \varepsilon$ (Remark 1 (iv)). Any extra daylighting must come from outside the block. Put bluntly, more houses must be built on the block's frontage. We define the subset of frontage vertices $V_o$, $V_o = \{(j,k) : f_{jk} \neq 0\}$. Vertex



(a) **chess block on lattice graph**
min. yard cover
$|C^c| = 15$

(b) **chess block on lattice graph**
max. matching
$|M| = 15$

(c) **chess block on frontage subgraph**

$|C_o| = 9$

(d) **alt. block $\boldsymbol{x}$ on frontage subgraph**

$|X_o| = 11$

Notes: (i) Panel (a) shows bipartition $\boldsymbol{b}$, coincident with chess block $\boldsymbol{c}$, on the (block) graph $G$. (ii) Panel (b) shows a minimum yard cover (white vertices) as well as a maximum matching (heavy independent edges). (iii) Panel (c) restricts the chess block to the block's boundary. (iv) Panel (d) illustrates how two daylightings are lost if there are only 7 yards on the boundary.

Figure 5: Lattice graph $G$ and frontage subgraph $G_o$

set $V_o$ induces the *frontage subgraph* $G_o$ (Fig. (5c)). It has $\nu_o = 2(m + n - 2)$ vertices. Subgraph $G_o$ highlights the – internal – daylightings possible along the street frontage. Subgraph $G_o$ is a very simple graph. It is a *cycle*; each of its vertices has degree 2.

Now adding $\delta$ external daylightings means foregoing $\delta/2$ frontage yards at least. A block with $\nu_o/2 - \delta/2$ yards can provide at most twice as many internal daylightings in $G_o$. So having $\delta$ external daylightings more destroys $\delta$ internal daylightings in $G_o$ – and hence in $G$. We conclude that the chess block really *does* maximize daylit windows. Inequality (12) above becomes binding,

$$\max_{\boldsymbol{x}} \; \Lambda(\boldsymbol{x}) \;=\; \Lambda(\boldsymbol{c}) \;=\; \begin{cases} 2\,\nu & \text{if } \nu \text{ is even} \\ 2(\nu+1) & \text{if } \nu \text{ is odd.} \end{cases} \tag{13}$$

Fig. (5d) illustrates this. Its design has 7 frontage yards, which can daylight 14 windows at most. Yet there are 18 windows (one per each of the 18 edges) to be potentially daylit in $G_o$. So this design fails to turn on 4 of the frontage subgraph's, and hence also of the graph's, edges. The chess block maximizes daylit windows (Proposition 2 (ii)).

When $\nu$ is even, there are multiple maximizers. Then the chess block is not alone in maximizing daylit windows. Slight variations in design achieve maximum daylit windows, too. For example, consider adding to the chess block a house on one of its vacant corner lots, or on both vacant corner lots. This gives daylit windows maximizers that differ from chess block types $\boldsymbol{a}$ and $\boldsymbol{b}$. When $\nu$ is odd, the chess block is the only maximizer (Proposition 2 (iii)), however. Here the chess block already builds up on all four corner lots. To increase daylit windows further, any alternative design must import daylight by building up on a frontage lot. Building up on a frontage lot increases externally daylit windows by 1. But it also destroys internally daylit windows by 2 at least. No daylit windows can be gained by further building up on the block's frontage.

Of course, if the chess block maximizes daylit windows, it also maximizes daylit windows for when $N$ is constrained to equal $|C|$. So $|C|, 4|C|)$ in $(N, \Lambda)$-space is a first point on the daylight frontier (Proposition 2 (iv)). Further points on the frontier follow. Any number $\delta$ of houses less than $|C|$ deletes 4 windows that, since no occlusions with the chess block exist, were all daylit. Ignoring the integer constraint, the daylight frontier obeys the equation $\Lambda(N) = 4N$ for as long as $N$ is less than $|C|$ (Proposition 2 (v)).

**Proposition 2 (Daylight Frontier)**

(i) *Chess block internally daylit windows are max., $\Lambda_i(\boldsymbol{c}) = \max_{\boldsymbol{x}} \Lambda_i(\boldsymbol{x})$, and equal to $\varepsilon$.*
(ii) *Chess block daylit windows are max., $\Lambda(\boldsymbol{c}) = \max_{\boldsymbol{x}} \Lambda(\boldsymbol{x})$, and equal to $4|C|$.*
(iii) *For $\nu$ odd, maximum daylit windows imply the chess block, $\Lambda(\boldsymbol{c}) > \Lambda(\boldsymbol{x})$ for $\boldsymbol{x} \neq \boldsymbol{c}$.*
(iv) *Point $(|C|, 4|C|) = P$ satisfies the graph of the daylight frontier $(N, \Lambda(N))$.*
(v) *The daylight frontier is $\Lambda(N) = 4N$ for all integer $N$ between 0 and $|C|$.*

**Proof of Proposition 2**:

(i) Follows from Remark 1 (iv). □

(ii) Let there be some configuration $\boldsymbol{x} \neq \boldsymbol{c}$ such that $\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{c}) > 0$. But then $\Lambda_o(\boldsymbol{x}) - \Lambda_o(\boldsymbol{c}) > 0$. Define $\delta = \Lambda_o(\boldsymbol{x}) - \Lambda_o(\boldsymbol{c})$. To collect $\delta$ extra external daylightings, $\boldsymbol{x}$ must have at least $\delta/2$ streetfront house vertices more (and hence $\delta/2$ streetfront yard vertices less) than $\boldsymbol{c}$. Yet then only $2(\nu_o/2 - \delta/2)$ or $\nu_o - \delta$ edges of $G_o$, and hence of $G$, can be active at most. This is $\delta$ fewer active edges than with $\boldsymbol{c}$. Put differently,

$$\Lambda_o(\boldsymbol{x}) \;-\; \Lambda_o(\boldsymbol{c}) \;=\; \delta \;\leqslant\; \Lambda_i(\boldsymbol{c}) \;-\; \Lambda_i(\boldsymbol{x}) \tag{14}$$

and hence $\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{c}) \leqslant 0$. This is a contradiction. □

(iii) Consider some $\boldsymbol{x} \neq \boldsymbol{c}$ with $\Lambda(\boldsymbol{x}) = \Lambda(\boldsymbol{c})$. Via inequality (14), $\Lambda_o(\boldsymbol{x}) \leqslant \Lambda_o(\boldsymbol{c})$, else $\Lambda(\boldsymbol{x}) < \Lambda(\boldsymbol{c})$. Moreover, $\Lambda_o(\boldsymbol{x}) \geqslant \Lambda_o(\boldsymbol{c})$, else $\Lambda(\boldsymbol{x}) < \Lambda(\boldsymbol{c})$. Hence $\Lambda_o(\boldsymbol{x}) = \Lambda_o(\boldsymbol{c})$, which in turn implies $\Lambda_i(\boldsymbol{x}) = \Lambda_i(\boldsymbol{c}) = \varepsilon$, too. Thus $(X, X^c)$ is bipartite. So $\boldsymbol{x}$ must equal either $\boldsymbol{b}$ or $\boldsymbol{a}$. We may rule out $\boldsymbol{a}$ because $\nu$ is odd. □

(iv) Since $\Lambda(\boldsymbol{c}) \geqslant \Lambda(\boldsymbol{x})$ for all $\boldsymbol{x}$ (Proposition 3 (i)) ("globally"), it must be true that $\Lambda(\boldsymbol{c}) \geqslant \Lambda(\boldsymbol{x})$ for all $\boldsymbol{x}$ constrained to satisfy $N = \boldsymbol{\iota}'\boldsymbol{x} = |C|$ ("locally"). □

Fig. (6) shows the block's daylight frontier as the upper boundary of the shaded set. The shaded set is the block developer's convex opportunity set, and point $P$ is an extreme point of it. The figure also shows three contours of block profit $\Pi$, or $pN + v\Lambda$. Clearly these contours select vertex $P$ as the optimum design choice; i.e. they select the chess block. The chess block is the block developer's profit maximizing design (Proposition 3 (i)), and her maximum profit simply becomes $|C|(p + 4v)$ (Proposition 3 (ii)).

**Proposition 3 (The Chess Block Maximizes Block Profit)**

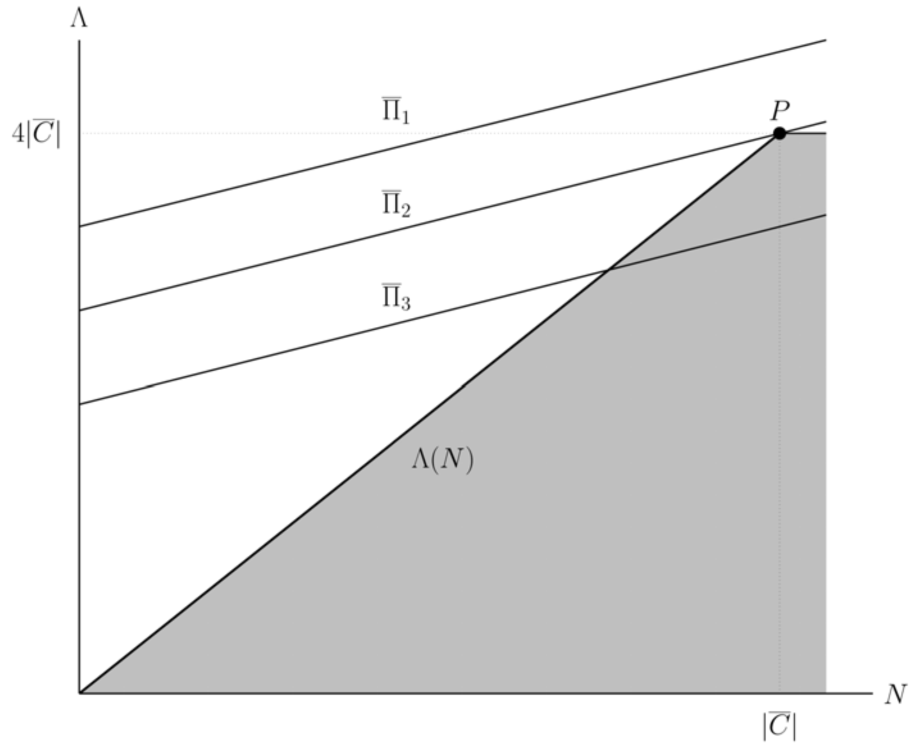
(i) *The chess block maximizes block profit, $\max_{\boldsymbol{x}} \Pi(\boldsymbol{x}) = \Pi(\boldsymbol{c})$.*
(ii) *Maximum profit function $\Pi(p, v)$* is $|C|(p + 4v)$.

**Proof of Proposition 3**:

(i) Suppose a configuration $\boldsymbol{x} \neq \boldsymbol{c}$ exists with strictly greater profit, $\Pi(\boldsymbol{x}) > \Pi(\boldsymbol{c})$. So $\boldsymbol{x}$ has strictly more houses (Proposition 2 (iv)). Yet it has no more daylit windows (Proposition 3 (i)). As $p \leqslant 0$ by eq. (1), $\Pi(\boldsymbol{x}) \leqslant \Pi(\boldsymbol{c})$. □

(ii) Each of the chess block's $|C|$ houses is daylit fully, and has value $p + 4v$. □

Moreover, the chess block even is the block developer's *only* profit maximizing design. The chess block alone (up to Definition 1's small ambiguity when block size $\nu$ is even)




Notes: (i) It is $\Lambda(N)$ the daylight frontier for when $\nu$ is even. (ii) Three profit contours are drawn, at profits $\overline{\Pi}_1$, $\overline{\Pi}_2$ and $\overline{\Pi}_3$. (iii) Vertex $P = (|C|, 4|C|)$ corresponds to the chess block. (iv) The slope of any profit contour is $-p/v \geqslant 0$. (v) The shaded set is $P$, which contains the block developer's daylight-houses opportunity set $\{(N, \Lambda) : \Lambda \leqslant \Lambda(N)\}$. (vi) Changes in $p$ or $v$ have no effect on the solution.

Figure 6: Daylight frontier and the chess block

sets out the optimum combination of houses and daylight (Proposition 4 (i)). Fig. (6) thus also motivates the chess block's ubiquity. Profit for blocks closer to the center exhibits larger $-p/v$. For those blocks, contours are steeper. Yet they still select vertex $P$. The chess block is the optimum choice at *any* distance $r < \widetilde{r}$ (Proposition 4 (ii)).

Suppose, for the moment, that all blocks in all rings in all cities are built by block developers. Then theirs will be an urban system of a highly repetitive character. With no exception, and at any distance from the city center, all blocks are chess blocks. One type of variation arises when $\nu$ is even. Then the two designs $\boldsymbol{b}$ or $\boldsymbol{a}$ may coincide. The only other (and trivial) type of variation obtains if blocks vary in size.

**Proposition 4 (With Block Developers, Chess Blocks are Ubiquitous)**

(i) *The chess block is the unique maximizer of block profit,* $\Pi(\boldsymbol{x}) < \Pi(\boldsymbol{c})$ *for all* $\boldsymbol{x}$.

(ii) *The chess block almost always, i.e. in all rings* $r < \tilde{r}$, *is the unique maximizer.*

**Proof of Proposition 4**:

(i) Point $P = (|C|, 4|C|)$ is an extreme point of the convex constraint set. □

(iv) By eq. (1), $p$ is strictly decreasing in $r$, and so profit contours' slope $-p/v$ is strictly decreasing in $r$, too. □

This is our first foray into the chess block. We have derived a number of the chess block's fundamental daylighting properties. In particular, we have *explained* the chess block. The chess block becomes the outcome of a block developer's profit motive, rather a merely remarkable pattern. Economically, the chess block really is both: an exceedingly simple design principle and a recipe for maximum profit. This coincidence may contribute to explaining the chess block's growing adoption. We reveal further chess block properties shortly. The chess block emerges as a Nash-equilibrium configuration when sub-developers of the block tacitly collude in the city periphery (Proposition 5). Moreover, the chess block turns out to "almost" have the minimum number of houses necessary to comprehensively shade the block's yards (Proposition 10).

Before we turn to the analysis of sub-developer equilibrium, we note that while we have cast our analysis in terms of a rectangular block (where notation is simpler), these propositions do not hinge on the rectangular layout. *Any* layout of the block gives rise to the results in Proposition 1 through 4 as long as the corresponding lattice graph has minimum degree of 2. The proof obtains by retracing those propositions' proofs for that general layout.

**Note (Chess Blocks are Built on Irregular Blocks, Too)**
*Propositions 1 through 4 extend to any block whose lattice graph has minimum degree 2.*

# 4 The Chess Estate

Historically it is not the block developer who single-handedly and seamlessly develops the entire block. Instead *sub-developers* develop subdivisions of the blocks, or *estates*. We now explore the equilibrium spatial structure if decisions on estates are taken by sub-developers. We set out the resulting mesh of houses and yards, joint with their attendant daylightings and occlusions, at any distance from the city center. Near it, so we show first, sub-developers partly occlude each other. Resulting equilibrium blocks are inefficient. After all, they do not replicate the only profit maximizing design, i.e. the chess block. But these equilibrium blocks also provide more (and denser, darker) housing than the chess block (Proposition 5). Further out from the city center, so we show then, sub-developers tacitly collude in not occluding each other. Now equilibrium blocks are both efficient and less dense (Proposition 6). We solve for the urban spatial structure that accommodates the closed city's given population (Proposition 7).

So let us first partition the block into $S \geqslant 2$ subsets indexed by $s = 1, \ldots, S$. These are *estates*. Any estate is such that any of its lots has two neighbors within that estate. We do not require an estate's parcels to be contiguous. A sub-developer's estate may consist of disconnected subsets. Then the sub-developer optimizes separately on each subset. To address the lots of estate $s$, we introduce the $n \times 1$ ownership dummy $\boldsymbol{\iota}_s$ featuring 1's for all lots belonging to estate $s$ and 0 entries everywhere else. Via aggregation, $\sum_{s=1}^{S} \boldsymbol{\iota}_s$ yields $\boldsymbol{\iota}$. The vertex set of estate $s$ is $V_s = \{(j,k) : \iota_{s,jk} \neq 0\}$. It induces the estate's subgraph $G_s = (V_s, E_s)$ with $\nu_s = |V_s|$ vertices and $\varepsilon_s = |E_s|$ edges.

We trace *estate design* vector $\boldsymbol{x}_s$ by

$$x_{s,jk} = \begin{cases} 0 & \text{if} \quad jk & & \text{is not owned by } s \\ 1 & \text{if} \quad jk \text{ has a house} & \text{and} & \text{is owned by } s \\ 0 & \text{if} \quad jk \text{ is a yard} & \text{and} & \text{is owned by } s \end{cases} \qquad \text{(estate design } \boldsymbol{x}_s\text{)}, \quad (15)$$

where $x_{s,jk}$ is the sub-developer's construction decision on lot $jk$. This completes our description of estate design $\boldsymbol{x}_s$. By aggregation, $\sum_{s=1}^{S} \boldsymbol{x}_s$ yields $\boldsymbol{x}$, the composite configuration of the block. Both types of notation, i.e. $\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_S$ and $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_S)$, capture the configuration produced by combining sub-developers' sub-designs. We also define $\boldsymbol{x}_{-s} = \boldsymbol{x} - \boldsymbol{x}_s$ and dummy $\boldsymbol{\iota}_{-s} = \boldsymbol{\iota} - \boldsymbol{\iota}_s$. It is $\boldsymbol{x}_{-s}$ the vector of design decisions taken by all sub-developers other than $s$, with entries for sub-developer $s$'s lots set to 0.

Much as with the entire block, the estate's internally daylit windows $\Lambda_{s,i}(\boldsymbol{x}_s)$ are its windows daylit from within estate $s$. The estate's externally daylit windows $\Lambda_{s,o}(\boldsymbol{x})$, however, no longer only count windows daylit by the street; they also count windows daylit by neighboring estates' neighboring yards. In short,

$$\Lambda_{s,i}(\boldsymbol{x}_s) \quad = \quad \boldsymbol{x}_s'\boldsymbol{A}(\boldsymbol{\iota}_s - \boldsymbol{x}_s) \qquad \text{(estate internally daylit windows)} \quad (16)$$

$$\Lambda_{s,o}(\boldsymbol{x}_s) \quad = \quad \boldsymbol{x}_s'\boldsymbol{f} \; + \; \boldsymbol{x}_s'\boldsymbol{A}(\boldsymbol{\iota}_{-s} - \boldsymbol{x}_{-s}) \quad \text{(estate externally daylit windows).} \quad (17)$$

Now the sub-developer's daylit windows, $\Lambda_s$, become

$$\Lambda_s(\boldsymbol{x}_s) \quad = \quad \Lambda_{s,i}(\boldsymbol{x}_s) \; + \; \Lambda_{s,o}(\boldsymbol{x}_s) \qquad \text{(estate daylit windows)} \quad (18)$$

$$\Lambda_s(\boldsymbol{x}_s) \quad = \quad 4\boldsymbol{\iota}'\boldsymbol{x}_s - \; \boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_s \; - \; \boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_{-s} \qquad \text{(estate daylit windows),} \quad (19)$$

from either a compositional or a residual perspective, respectively. Here the term $\boldsymbol{x}_s'\boldsymbol{A}(\boldsymbol{\iota}_{-s} - \boldsymbol{x}_{-s})$ captures the (positive) inter-estate daylighting externalities; while the term $\boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_{-s}$ highlights the (negative) inter-estate occlusion externalities.

**Remark 4 (Bipartite Sub-Designs)**

(i) *Sub-graph $G_s$ is bipartite, and its bipartite sub-designs are $\boldsymbol{b}_s$ and $\boldsymbol{a}_s$.*

(ii) *Bipartite sub-designs' internally daylit windows are maximum, $\Lambda_{i,s}(\boldsymbol{b}_s) = \Lambda_{i,s}(\boldsymbol{a}_s) = \varepsilon_s$.*

(iii) *Facing $\boldsymbol{b}_{-s}$, sub-design $\boldsymbol{b}_s$ has $4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$ more (externally) daylit windows than $\boldsymbol{a}_s$.*

**Proof**: (i) Consider any edge in the sub-graph. By definition, that edge also is an edge in the block graph. Because the graph is bipartite, the edge links to both a house and a yard. So the sub-graph is bipartite, too. □

(ii) Similar to the proof of Remark 1 (iv). □

(iii) Absent a neighboring estate's occlusions, $\boldsymbol{b}_s$ has 4 externally daylit windows more than $\boldsymbol{a}_s$. (It builds up on corner lots.) Now subtract $\boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{x}_{-s}$ occlusions. □

We next represent estate $s$ by its induced *sub-graph*. This is the subset of vertices corresponding to the estate's lots $V_s$ joint with the subset of edges $E_s$ linking any pair of those vertices in the block's original graph $G$. So $G_s = (V_s, E_s)$. Sub-graph $G_s$ has $\nu_s$ vertices and $\varepsilon_s$ edges. An important property of sub-graph $G_s$ is its bipartiteness; this it inherits from its parent graph $G$ (Remark 4 (i)). Now let us label the two partite sets of that bipartition $B_s$ and $B_s^c$ such that $|B_s| \geqslant |B_s^c|$. Set $B_s$ defines sub-design $\boldsymbol{b}_s$ (the $mn \times 1$ sub-design vector has 1's whenever the corresponding lot belongs to $B_s$ and 0's everywhere else), and $\boldsymbol{a}_s = \boldsymbol{\iota}_s - \boldsymbol{b}_s$. Both $\boldsymbol{b}_s$ and $\boldsymbol{a}_s$ maximize windows daylit from within the estate (Remark 4 (ii)). However, $\boldsymbol{b}_s$ has 1 house, and when faced with $\boldsymbol{x}_{-s}$ also $4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{x}_{-s}$ externally daylit windows, more than $\boldsymbol{a}_s$ (Remark 4 (iii)).

We next define the *chess estate*. If $\nu$ is odd, the chess estate coincides with bipartite design $\boldsymbol{b}_s$ (Definition 2). The chess estate is for the estate what the chess block is for the block, i.e. a bi-coloring layout of houses and yards.

**Definition 2 (Chess Estate)**

*The chess estate $\boldsymbol{c}_s$ equals $\boldsymbol{a}_s$ or $\boldsymbol{b}_s$ if $\nu_s$ is even – and $\boldsymbol{b}_s$ if $\nu_s$ is odd.*

Blocks can be partitioned into estates in many different ways. We begin by dividing the rectangular block into two identical estates along its horizontal axis of symmetry. For now we restrict attention to blocks that are "small", i.e. for which $n \leqslant 5$. Fig. (7) illustrates one such symmetric partition of a small block into two equal-sized estates, on the $6 \times 5$-block. Lots on one side of the inter-estate boundary (heavy dividing line) belong to one sub-developer, lots on the other side to the other. Much can be learnt from consulting this special case. But neither of subsequent Propositions 5 and 6 are restricted to the $6 \times 5$ block. Our results go through unchanged also with two symmetric odd-sized estates on, say, the $6 \times 3$-block, the $6 \times 1$-block, the $2 \times 3$-block, etc. In any case, the problem of finding a Nash-equilibrium remains formidable enough with a $6 \times 5$ -block. Either sub-developer has $2^{15}$ sub-designs to choose from.

The sub-developer maximizes subdivision profit, $\Pi_s$. So he solves

$$\max_{\boldsymbol{x}_s, N_s} \quad \Pi_s = pN_s + v\big(4N_s - \boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_s - \boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_{-s}\big) \text{ s.t. } \boldsymbol{\iota}'\boldsymbol{x}_s = N_s \text{ and } x_{s,jk} \in \{0,1\}. \quad (20)$$

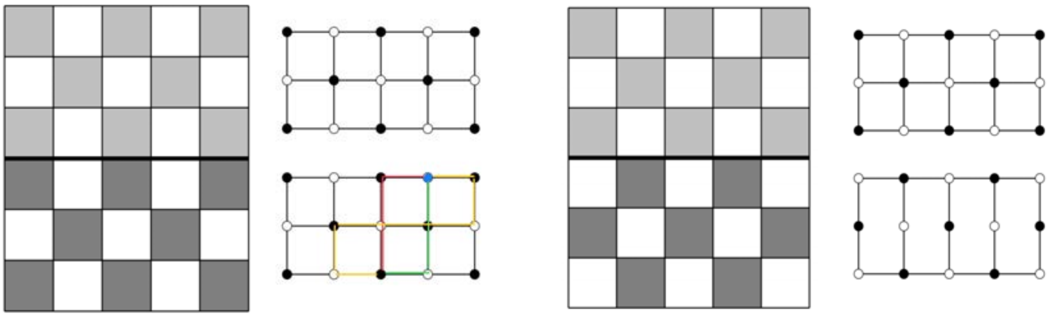
Earlier we saw how block profit maximization can be broken down into steps. Essentially that same procedure we may apply to the estate, too. In the first step we identify the estate's daylight frontier $\Lambda(N_s, \boldsymbol{b}_{-s})$, conditional on the other estate's sub-design $\boldsymbol{b}_{-s}$ (eq. (21)). In the second step we pick the profit maximizing point on that frontier (eq. (22)). The only novelty here is that estate frontiers are interdependent. One estate's frontier depends on the other estate's design, and vice versa. This interdependence shows up in the composition of the estate's externally daylit windows, $\Lambda_{s,o}$ (see eq. (17)). Those, after all, directly depend on the neighboring estate's design, $\boldsymbol{x}_{-s}$.

$$\begin{aligned} \Lambda_s(N_s, \boldsymbol{b}_{-s}) &= \max_{\boldsymbol{x}_s} \ \Lambda_{s,i}(\boldsymbol{x}_s) + \Lambda_{s,o}(\boldsymbol{x}_s) \text{ s.t. } \boldsymbol{\iota}'\boldsymbol{x}_s = N_s \text{ (estate frontier).} \quad (21) \\ \Pi_s(p, v) &= \max_{N_s} \ pN_s + v\Lambda_s(N_s, \boldsymbol{b}_{-s}) \quad \text{(max. estate profit).} (22) \end{aligned}$$

The first step calls for finding either estate's daylight frontier, $\Lambda_s(N_s, \boldsymbol{b}_{-s})$, i.e. the daylight frontier implied by the other estate's choosing the chess estate design $\boldsymbol{b}_s$. We start by identifying that point on the frontier that yields maximum daylit windows for the estate. This is the combination of houses and daylit windows associated with the chess estate $\boldsymbol{b}_s$ (Proposition 5 (i)). Chess estate $\boldsymbol{b}_s$ fully exploits the estate's capacity for internal daylighting; it then adds the most external daylight possible, even as it also incurs $\boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$ occlusions. The sub-developer could minimize those, by shifting occluded houses along the inter-estate boundary into more advantageous positions.

Unfortunately, such rearrangements on frontage lots, away from the pattern dictated by the chess estate, destroy more internal daylightings than they add external ones. In Fig. (7a), shifting the occluded house on lot $(4, 3)$ in $B$ to lot $(4, 4)$ in $B^c$ creates a single external daylighting at the expense of three internal daylightings, lost along each of the three odd-length paths in blue, green and yellow on screen. The remainder of the daylight frontier then is straightforward. To the left of $Q$, it has slope 4, and to the right of $Q$ it has slope $4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s} < 4$ (Fig. (8a)). The daylight frontier has an extreme point, or a "kink", at $Q$ (Proposition 5 (iii)). It defines the (shaded) convex opportunity set.

In the second step the sub-developer selects the best combination of houses and daylight on the frontier (point $R$ in Fig. (8a)). As long as contours slope upwards by strictly



(a) **Chess Estates Block $(\boldsymbol{b}_1, \boldsymbol{b}_2)$**
*iff* $0 \leqslant -p/v < 1$

$$N_1 = 8 = N_2$$
$$\Lambda_{1,i} = 22 = \Lambda_{2,i}$$
$$\boldsymbol{b}_1'\boldsymbol{A}\boldsymbol{b}_2 = 3 = \boldsymbol{b}_2'\boldsymbol{A}\boldsymbol{b}_1$$
$$\boldsymbol{b}_1'\boldsymbol{A}(\boldsymbol{\iota}_2 - \boldsymbol{b}_2) = 0 = \boldsymbol{b}_2'\boldsymbol{A}(\boldsymbol{\iota}_1 - \boldsymbol{b}_1)$$
$$\boldsymbol{b}_1'\boldsymbol{f} = 7 = \boldsymbol{b}_2'\boldsymbol{f}$$
$$\Lambda_1 = 29 = \Lambda_2$$
$$\Lambda = 58$$

(b) **Chess Block $(\boldsymbol{a}_1, \boldsymbol{b}_2)$**
*iff* $1 < -p/v < 4$

$$N_1 = 7,\ N_2 = 8$$
$$\boldsymbol{a}_1'\boldsymbol{A}(\boldsymbol{\iota}_1 - \boldsymbol{a}_1) = 22,\ \boldsymbol{b}_2'\boldsymbol{A}(\boldsymbol{\iota}_2 - \boldsymbol{b}_2) = 22$$
$$\boldsymbol{a}_1'\boldsymbol{A}\boldsymbol{b}_2 = 0,\ \boldsymbol{b}_2'\boldsymbol{A}\boldsymbol{a}_1 = 0$$
$$\boldsymbol{a}_1'\boldsymbol{A}(\boldsymbol{\iota}_2 - \boldsymbol{b}_2) = 2,\ \boldsymbol{b}_2'\boldsymbol{A}(\boldsymbol{\iota}_1 - \boldsymbol{a}_1) = 3$$
$$\boldsymbol{a}_1'\boldsymbol{f} = 4\,,\ \boldsymbol{b}_2'\boldsymbol{f} = 7$$
$$\Lambda_1 = 28,\ \Lambda_2 = 32$$
$$\Lambda = 60$$

Notes: (i) Here the block is divided into two equal-sized estates. Estate 1 comprises all (black or white) lots below the horizontal line, estate 2 all (grey or white) lots above it. (ii) Expressions in the captions successively document estates': total houses ($\boldsymbol{\iota}'\boldsymbol{x}_s$), within-block daylightings ($\boldsymbol{x}_s'\boldsymbol{A}(\boldsymbol{\iota}_s - \boldsymbol{x}_s)$), cross-block occlusions or negative externalities ($\boldsymbol{x}_s'\boldsymbol{A}\boldsymbol{x}_{-s}$), cross-block daylightings or positive externalities ($\boldsymbol{x}_s'\boldsymbol{A}(\boldsymbol{\iota}_{-s} - \boldsymbol{x}_{-s})$), street daylighting ($\boldsymbol{x}_s'\boldsymbol{f}$) and daylit windows $\Lambda_s$. (iii) Panel (a) shows two chess estates, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Panel (b) has a chess estate for estate 2, $\boldsymbol{c}_2$, but not for estate 1.

Figure 7: Nash-equilibria on two estates

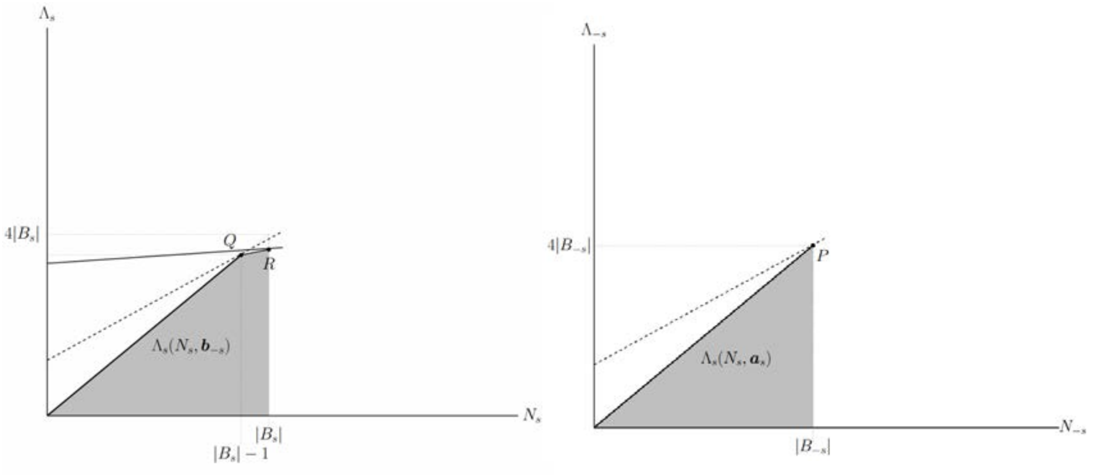
less than 1 (i.e. the unbroken straight contour in Fig. (8a)), the profit maximizing design also is the one that has maximum daylight windows on the estate. Thus the chess estate is the optimal response to the opposing estate's chess estate, and vice versa. Daylight frontier and optimum solution shown in Fig. (8a) really apply to both players, in setting $s$ equal to 1 *and* 2. A symmetric Nash-equilibrium $(\boldsymbol{b}_s, \boldsymbol{b}_s)$ has both sub-developers implement the chess estate (Proposition 5 (iv)). Note the commentary to Fig. (7a). It fills in a number of details we have not addressed in the main text.

Let us inspect the block configuration $\boldsymbol{b}_1 + \boldsymbol{b}_2$ that obtains in Proposition 5's Nash-equilibrium. This configuration clearly differs from the chess block $\boldsymbol{c}$ (in either of the latter's appearances). Not only does it have occlusions (which the chess block has not). Also it has 1 house more (than the chess block). Nash-equilibrium $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ is *denser* than the chess block $\boldsymbol{c}$. Sub-developers in Fig. (7a) do what we expect them to do: they ignore the occlusion externality from building up, and so build up more than they otherwise would. As a result, Nash-equilibrium is inefficient (Proposition 5 (v)).

**Proposition 5 (Equilibrium: Chess Estates Blocks near the Center)**
*Suppose $m \geqslant 2$, $n \leqslant 6$, and $\nu_s$ odd.*

(i) *The estate sub-graph $G_s$ has edge connectivity 2, $\kappa_s = 2$.*
(ii) *The chess estate $\boldsymbol{b}_s$ maximizes daylit windows*, $\max_{\boldsymbol{x}_s} \Lambda_s(\boldsymbol{x}_s) = \Lambda_s(\boldsymbol{b}_s) = 4|B_s| - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$.
(iii) *The estate's daylight frontier $\Lambda_s(N_s, \boldsymbol{b}_{-s})$ is*

$$\Lambda_s(N_s, \boldsymbol{b}_{-s}) = \begin{cases} 4N_s & \text{for all integers between 0 and } |B_s| - 1. \\ \Lambda_s(\boldsymbol{b}_s) & \text{for } N_s = |B_s| \end{cases} \tag{23}$$




(a) Facing a chess estate $\boldsymbol{b}_{-s}$

(b) Facing bipartite sub-design $\boldsymbol{a}_s$

Figure 8: Estates' daylight frontiers and optimal actions in Nash-Equilibrium

(iv) *A profile of chess estates, $(\boldsymbol{b}_s, \boldsymbol{b}_{-s})$, is a Nash-equilibrium iff $-p/v < 4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$.*

(v) *Nash-equilibrium $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ has 1 house more than the chess block $\boldsymbol{c}$, and is inefficient.*

**Proof of Proposition 5**:

(i) Two vertices of the lattice graph $G_s$ in Fig. (7) lie on a common cycle. $\square$

(ii) Consider some $\boldsymbol{x}_s \neq \boldsymbol{b}_s$ such that $\delta_s = \Lambda_s(\boldsymbol{x}_s) - \Lambda_s(\boldsymbol{b}_s) > 0$. If $\boldsymbol{x}_s$ has more (or less) frontage houses than $\boldsymbol{b}_s$, then $\Lambda(\boldsymbol{x}_s) < \Lambda(\boldsymbol{b}_s)$ (akin to Proposition 2 (ii)). So $\boldsymbol{x}_s$ has as many frontage houses as $\boldsymbol{b}_s$. But then $\delta_s \leqslant 2$. (Only 2 frontage houses are not daylit with $\boldsymbol{b}_s$.) To collect those 2 extra external daylightings, $\boldsymbol{x}_s$ must surely place (at least) 1 house on a lot in $B_s^c$ and another on a lot in $B_s$. There are 2 edge-disjoint paths between those houses (by Proposition 5 (i)). These are of odd length (by Remark 1 (v)), and so 2 internal daylightings are lost. Thus

$$\Lambda_{s,o}(\boldsymbol{x}_s) \;-\; \Lambda_{s,o}(\boldsymbol{b}) \;=\; 2 \;\leqslant\; \Lambda_{s,i}(\boldsymbol{b}_s) \;-\; \Lambda_{s,i}(\boldsymbol{x}_s), \tag{24}$$

and so $\Lambda_s(\boldsymbol{x}_s) - \Lambda_s(\boldsymbol{b}_s) \leqslant 0$. $\square$

(iii) At $N_s < |B_s|$, $4N_s$ is the maximum number of daylit windows (by eq. (19)); while at $N_s = |B_s|$, $\Lambda_s(\boldsymbol{b}_s) = 4|B_s| - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$ is (by Proposition 5 (ii))). $\square$

(iv) Profit contours for estate $s$ have slope $-p/v$. Since by assumption $-p/v$ is less than the slope of the daylight frontier "after the kink", $4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$, the highest contour selects point $Q$ on the estate's frontier (Fig. (8a)). Sub-design $\boldsymbol{b}_s$ is the best response to chess estate $\boldsymbol{b}_{-s}$. Now apply $s = 1, 2$ to parts (i), (ii), (iii). $\square$

(v) Since $\boldsymbol{\iota}'(\boldsymbol{b}_1 + \boldsymbol{b}_2) = \boldsymbol{\iota}'\boldsymbol{b}_1 + \boldsymbol{\iota}'\boldsymbol{a}_2 + 1$, the equilibrium block $\boldsymbol{b}_1 + \boldsymbol{b}_2$ has 1 house more than the chess block $\boldsymbol{b}_1 + \boldsymbol{a_2}$, or $\boldsymbol{c}$. Also, profit $\Pi(\boldsymbol{b}_1 + \boldsymbol{b}_2)$ is not maximum because (by Proposition 4 (i)) uniquely profits $\Pi(\boldsymbol{b}_1 + \boldsymbol{a}_2)$ or $\Pi(\boldsymbol{a}_1 + \boldsymbol{b}_2)$ are. $\square$

Proposition 5 discusses Nash equilibrium if profit contours slope upwards by less than $4 - \boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$. Proposition 6 next addresses Nash-equilibrium for when the contours of profit slope upwards by more. For that Nash-equilibrium we suggest $(\boldsymbol{b}_1, \boldsymbol{a}_2)$ or $(\boldsymbol{a}_1, \boldsymbol{b}_2)$ as trial candidates. Fig. (7b) illustrates $(\boldsymbol{a}_1, \boldsymbol{b}_2)$, and Figs. (8a) and (8b) jointly illustrate the two sub-developers' asymmetric decisions. Sub-developer 2 (in Fig. (8b), for $s = 2$ and $-s = 1$), in facing $\boldsymbol{a}_1$, faces no occlusions when building chess estate $\boldsymbol{b}_2$.

This is his profit maximizing choice $P$ on the frontier, given the dashed contour.

**Proposition 6 (Equilibrium: Chess Blocks Away from the Center)**
*Suppose $m \geqslant 2$, $n \leqslant 6$, and $\nu_s$ odd.*

(i) *The frontier of estate $s$ when facing $\boldsymbol{b}_{-s}$ is given by eq. (23).*
(ii) *The frontier of estate $s$ when facing $\boldsymbol{a}_{-s}$ is $\Lambda_s(N_s, \boldsymbol{a}_{-s}) = 4N_s$ for all integers between 0 and $|B_s|$.*
(iii) *Strategy profiles $(\boldsymbol{b}_s, \boldsymbol{a}_{-s})$ and $(\boldsymbol{a}_s, \boldsymbol{b}_{-s})$ are Nash-equilibria iff $-p/v \geqslant 4 - \boldsymbol{b}_s' \boldsymbol{A} \boldsymbol{b}_{-s}$.*
(iv) *Both Nash-equilibria mimic the chess block. We say that sub-developers tacitly collude.*

**Proof of Proposition 6**:

(i) See proof of Proposition 5 (iii).

(ii) By Proposition 2 (ii), $4N_s$ is the global maximum. So it also is given $\boldsymbol{a}_{-s}$. □

(iii) Given $-p/v \geqslant 4 - \boldsymbol{b}_s' \boldsymbol{A} \boldsymbol{b}_{-s}$, profit contours are dashed lines in Figs. (8a) and (8b). Point $Q$ (i.e. sub-design $\boldsymbol{a}_s$) is best for sub-developer 1 given $\boldsymbol{b}_{-s}$, and point $P$ (the chess estate) $\boldsymbol{b}_{-s}$ is best for sub-developer 2 given $\boldsymbol{a}_s$. Apply $s = 1, 2$. □

(iv) It is $\boldsymbol{a}_1 + \boldsymbol{b}_2 = \boldsymbol{b}$ and $\boldsymbol{b}_1 + \boldsymbol{a}_2 = \boldsymbol{a}$. So the chess block $\boldsymbol{c}$ obtains always. □

Sub-developer 1 (in Fig. (8a, for $s = 1$ and $-s = 2$), given his opponent's $\boldsymbol{b}_2$, shies away from the occlusions involved when choosing the chess estate $\boldsymbol{b}_1$ also. Sub-developer 1 instead "plays" $\boldsymbol{a}_1$. Point $Q$ in Fig. (8a) yields his highest dashed contour. We note that the resulting aggregate configuration for the block, $\boldsymbol{b}_1 + \boldsymbol{a}_2$, coincides with the chess block. This identifies yet another property of the chess block. Now the chess block also emerges as an equilibrium outcome in sub-developer competition, too. Effectively sub-developers avoid confrontation. We say they *collude tacitly* (Proposition 6 (iv)). More succinctly, here sub-developers behave as if acting jointly and cooperatively, picking the design a block developer would pick (Propositions 1 through 4). More generally, developer counts do not capture real estate competition.

Propositions 5 and 6 determine the spatial structure of our closed city. Consider the expression $4 - \boldsymbol{b}_s' \boldsymbol{A} \boldsymbol{b}_{-s}$ first. On the one hand, expression gives the external daylighting advantage of the denser bipartite sub-design, $\boldsymbol{b}_s$, relative to $\boldsymbol{a}_s$. It represents the daylit windows to be gained from aggressively building up on the estate. On the other hand, the same expression is a threshold real rent. Whenever $-p/v$ falls short of it, chess estates block get built and sub-developers compete with, and occlude, each other; whereas whenever $-p/v$ exceeds it, chess blocks get built and sub-developers collude not to occlude. In Fig. (8a), we see how the chess estate block unravels when $-p/v$ falls. As the set of contours rotates downwards, point $R$ gets replaced by point $Q$. Not insisting on adding one more daylit window to the estate makes sense when this allows the sub-developer to avoid the risen cost $-p$ of having one house more.

Equating threshold dark rent with dark rent in eq. (1) and inverting for threshold distance $\widehat{r}$ gives the distance at which blocks switch designs,

$$\widehat{r} \ = \ \widetilde{r} \ - \ \boldsymbol{b}_s' \boldsymbol{A} \boldsymbol{b}_{-s} v/t \qquad \text{(threshold distance).} \qquad (25)$$

Developers supply different housing qualities to households (i.e. houses both with 4 and 3 daylit windows). Yet because any extra daylit window is priced at its valuation, households are indifferent between qualities. We may lump together the different qualities on offer into a single housing supply aggregate, which in turn accommodates exogenous city population $L$. So

$$\frac{\nu+1}{2}\sum_{\rho=1}^{\widetilde{r}/(m+1)} 4\big(1+r\big) + \sum_{\rho=1}^{\widehat{r}/(m+1)} 4\big(1+r\big) = L/h \quad \text{(housing market equilibrium).} \tag{26}$$

is the housing market equilibrium condition. Equations (25) and (26) jointly determine the equilibrium size of the city $\widetilde{r}$ and the threshold distance $\widehat{r}$. Proposition 7 summarizes the properties of a city that is built by pairs of sub-developers on odd-sized estates.

**Proposition 7 (Equilibrium Urban Spatial Structure with Sub-Developers)**
*Suppose $m \geqslant 2$, $n \leqslant 6$ and all $\nu_s$ odd.*

(i) *Chess estates $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ are built from 0 to $\widehat{r}$, and chess blocks $\boldsymbol{c}$ from $\widehat{r}$ out to $\widetilde{r}$.*
(ii) *Cities with larger blocks (i.e. $n$ larger) have chess blocks earlier, and more.*

**Proof of Proposition 7**:

(i) In main text.

(iii) Expression $\boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$ is increasing in $n$ and, by eq. (25), so is $\widetilde{r} - \widehat{r}$. But then $\widehat{r}$ is decreasing in $n$ (else housing supply on the l.h.s. of (26) exceeds $L$). □

We finally turn to the role of time-at-home for urban spatial structure and rents.

## 5 Time at Home

The secular reduction in working hours, and in time-at-work more recently, suggests (if not implies) that households spend more time at home. Once they do so, they should appreciate the quality of their home more. In particular, we should observe a secular increase in daylight valuation $v$. Let us experiment with how this affects city equilibrium.

Suppose $\widehat{r}$ were to increase (so that $\widetilde{r}$ were to increase, too). Then housing supply would exceed city population, violating (26). So threshold $\widehat{r}$ must decrease in response to an increasing $v$. At the same time, $\widetilde{r}$ must increase. Suppose it did not. Then housing supply would fall short of $L$, again violating (26). In sum, the city expands, is less dense, and occludes itself less. Also, the number of denser blocks contracts while pure chess blocks expand. Sub-developers collude more than ever (Proposition 7 (i)). Bright houses not only increase their share in city housing; they have become more expensive as $p(r) + 4v$ unambiguously increases in $\widetilde{r}$, via eq. (1) (Proposition 7 (ii)).

Ultimately, when $\widehat{r}$ has fallen to 0, only chess blocks remain. Dark houses may have become cheap; but they have also become unavailable. Dark houses have become crowded out. The only remaining housing quality are fully daylit houses, and these houses have become ever more expensive (Proposition 7 (iii)).

**Proposition 7 (Time at Home, Developer Collusion, and House Prices)**

(i) *Prevalence of chess blocks is increasing (of chess estates blocks is decreasing) in time-at-home.*
(ii) *Bright rent $p(r) + 4v$ (dark rent $p(r)$) is increasing (decreasing) in time-at-home.*
(iii) *Fully daylit houses are the only remaining quality once valuation $v$ exceeds $t\widetilde{r}/\boldsymbol{b}_s'\boldsymbol{A}\boldsymbol{b}_{-s}$.*

## 6 Conclusions

Anecdotally, real-world blocks increasingly resemble the chess block. We first show that the chess block maximizes daylit houses on the block, daylit windows on the block, and block profit. These properties explain the chess block's perennial appeal to large real estate developers. Moreover, these properties have growing appeal to real estate developers as households' appreciation of daylight grows. A secular reduction in working hours, but also more opportunities for working from home, suggest that today households value daylight more than they did in the past. Greater daylight valuation begets greater collusion in the real estate industry. Only colluding developers are able to fully internalize, and hence best exploit, the within-block externalities that characterize urban daylight. Collusion drives real estate prices up, by driving marginal residents out.

A large strand of the recent urban economics literature asserts that local government zoning prevents cities from becoming denser. This paper asserts that households value daylight, and that urban yards generate the daylight that households desire. If chess blocks are profit maximizing to today's large and small developers, building up on their yards will make property values decrease. This not only explains strong resistance against rezoning across the US; it also questions its welfare justification. Another strand of the recent urban economics literature asserts that urban land parcels may be too small for modern development. Land assembly may not prevail as much as it should. This paper asserts that large ("block") developers are even less likely than smaller ("sub-") developers to build up on their land. Small developers push for more housing than large developers do. More land assembly will create less dense, instead of denser, housing.

We note that the chess block is not just about daylight. It also provides its houses with the many amenities that typically come along with the availability of daylight. Homes that are daylit better also are more likely to: enjoy greater calm, have a view, be more private, extract photovoltaic energy, allow its residents to evade air-borne infectious diseases such as Covid-19, escape one's neighbors' emissions from cooking, smoking etc. Alternatively, homes that are daylit better may offer access to a "garden club" whose members are the residents of the houses adjacent to it. We suppress mention of these complementary benefits throughout this paper. But they may be just as relevant as, or even more relevant than, daylight itself is. They reinforce resident, developer and ultimately also our interest in the availability of urban daylight.

A companion paper of ours (Dascher/Haupt (2025)) goes beyond the question of optimal block design, exploring optimum urban design instead. With its broken streetfront, the chess block reduces the block's attention to the street. There plainly are fewer eyes on the street. Jacobs-style safety in public space suffers if streets' adjoining blocks no longer align with the street. In addition, the chess block offers less retail space aligning with the street. All of this makes the chess block's daylighting properties less impressive. There may be a trade-off to resolve between the quality of its private space and that of its public space. The chess block is not necessarily the optimal urban block.

Empirical work will have to contend with the endogeneity our model implies. If time-at-home drives up block uniformity, and if block uniformity really devastates urban retail and neighborhood safety, then block uniformity likely drives up time-at-home. Dynamically, increasing time-at-home becomes self-reinforcing. The more we stay at home, the less varied and safe our public spaces, and . . . hence the more we must stay at home.

Suppose time-at-home really does play the secular role we attribute to it here. We address two concerns for the near future. (1) (Return to Office) If time-at-home increases further (e.g. via an even shorter work week), so will the price of real estate. Alternatively, should time-at-home fall (as with a policy of return-to-office), so will the price of real estate. (2) (Global warming) If the planet heats up further, today's daylight valuation may well overestimate tomorrow's daylight valuation. Soon it may be shaded, rather than daylit, living that enjoys the positive premium. If positive dark rent enters the flow of discounted future rents, chess block proliferation will not serve us well.

# 7 Appendix

Replace $\Lambda_i(\boldsymbol{x}) + \Lambda_o(\boldsymbol{x})$ in maximization problem (9) by $\Lambda$, then use decomposition (7) and rewrite the resulting expression as

$$\max_N \left\{ \max_{\boldsymbol{x}} \; pN \; + \; v(4N \; - \; \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) \quad \text{s.t.} \quad x_{jk} \in \{0,1\} \;\; \text{and} \;\; \boldsymbol{\iota}'\boldsymbol{x} = N \right\}$$

Shifting $pN$ and $v$ (as constants to the inner maximization problem) outside the curly brackets gives

$$\max_N \quad pN \; + \; v \left\{ \max_{\boldsymbol{x}} (4N \; - \; \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) \quad \text{s.t.} \quad x_{jk} \in \{0,1\} \;\; \text{and} \;\; \boldsymbol{\iota}'\boldsymbol{x} = N \right\}$$

The maximization problem in curly brackets defines the daylight frontier, $\Lambda(N)$, in eq. (10), and the resulting overall program becomes that given in (11).

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